

Capital Asset Pricing Model (CAPM) with CARA Utility and Normal Returns

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0. Introduction

The capital asset pricing model (CAPM) prices each risky asset by its beta against the market portfolio. This note derives it from investors with CARA utility and normal returns in four steps: [solve one investor's portfolio problem](#), rescale the solution into the [super-efficient portfolio](#), add up all investors' [demands](#) so that the super-efficient portfolio becomes the [market portfolio](#), and read the [beta relation](#) off the [market-clearing condition](#).

1. Setup and Optimization Problem

(i) The Portfolio

Suppose we hold risky assets with weights $\theta \in \mathbb{R}^N$ and the risk-free asset with weight $\theta_f \in \mathbb{R}$, where $\theta' \mathbf{1} + \theta_f = 1$ and the starting wealth is normalized to 1. Here $\mathbf{1} \in \mathbb{R}^N$ is the all-ones vector. Denote by $R \in \mathbb{R}^N$ the return vector of the risky assets, where $R \sim N(\mathbb{E}[R], \Sigma)$, and by R_f the return of the risk-free asset. The end-of-period wealth W of the [one unit](#) invested is

$$\begin{aligned} W &= \theta' R + \theta_f R_f \\ &= \theta' R + (1 - \theta' \mathbf{1}) R_f \\ &= R_f + \theta' (R - R_f \mathbf{1}). \end{aligned}$$

(ii) The Optimization Problem

Moreover, we consider a [CARA](#) utility function, $U(W) = -\exp(-aW)$, and what we want to do is to select θ and θ_f so that our expected utility is maximized. Therefore we have the following optimization problem.

$$\begin{aligned} & \max_{\theta, \theta_f} \mathbb{E}[U(W)] \\ &= \max_{\theta, \theta_f} \mathbb{E}[-\exp(-aW)]. \end{aligned}$$

As $R \sim N(\mathbb{E}[R], \Sigma)$ and W is an affine function of R , the **closure of the normal under affine maps** gives

$$W \sim N(R_f + \theta'(\mathbb{E}[R] - R_f \mathbf{1}), \theta' \Sigma \theta).$$

Therefore,

$$\mathbb{E}[-\exp(-aW)] = -\text{MGF}_W(-a)$$

where $\text{MGF}_W(\cdot)$ is the **moment generating function** of W .

As W follows the normal distribution, its MGF will be

$$\begin{aligned} -\text{MGF}_W(-a) &= -\exp\left(-\mathbb{E}[W]a + \frac{1}{2} \text{Var}[W]a^2\right) \\ &= -\exp\left(-a[R_f + \theta'(\mathbb{E}[R] - R_f \mathbf{1})] + \frac{a^2}{2} \theta' \Sigma \theta\right), \end{aligned}$$

so the objective function becomes

$$\begin{aligned} \mathbb{E}[U(W)] &= \mathbb{E}[-\exp(-aW)] \\ &= -\exp\left(-a[R_f + \theta'(\mathbb{E}[R] - R_f \mathbf{1})] + \frac{a^2}{2} \theta' \Sigma \theta\right). \end{aligned}$$

2. The Optimal Portfolio

(i) Solving the FOC

Write $\mu_e = \mathbb{E}[R] - R_f \mathbf{1}$ for the vector of expected excess returns, so the **objective** is $-\exp(g(\theta))$ with exponent

$$g(\theta) = -a(R_f + \theta' \mu_e) + \frac{a^2}{2} \theta' \Sigma \theta.$$

As $\exp$ is increasing, maximizing $-\exp(g)$ is the same as minimizing g . Dividing g by $-a < 0$ flips the minimization into a maximization, and dropping the constant R_f leaves

$$\max_{\theta} \theta' \mu_e - \frac{a}{2} \theta' \Sigma \theta.$$

Using $\nabla_{\theta}(\theta' \mu_e) = \mu_e$ and $\nabla_{\theta}(\theta' \Sigma \theta) = 2\Sigma\theta$ for symmetric Σ , the FOC is $\mu_e - a\Sigma\theta^* = 0$, so

$$\theta^* = \frac{1}{a} \Sigma^{-1} \mu_e = \frac{1}{a} \Sigma^{-1} (\mathbb{E}[R] - R_f \mathbf{1}).$$

Then the **budget constraint** gives the risk-free weight $\theta_f^* = 1 - \mathbf{1}'\theta^*$.

(ii) Second-Order Condition

This stationary point is the maximum. The Hessian of the objective is $-a\Sigma$, which is **negative definite** when Σ is positive definite. So the objective is strictly concave, and its stationary point is the unique global maximum.

3. The Super-Efficient Portfolio

This section focuses on portfolios of the risky assets only.

(i) Two-Fund Separation

Consider the portfolio $\theta_T \in \mathbb{R}^N$ that keeps the risky part of θ^* and drops the risk-free asset. It is derived by rescaling θ^* to satisfy the constraint $\theta_T' \mathbf{1} = 1$:

$$\begin{aligned} \theta_T &= \frac{1}{\mathbf{1}'\theta^*} \theta^* \\ &= \left(\frac{1}{a} \mathbf{1}' \Sigma^{-1} \mu_e \right)^{-1} \frac{1}{a} \Sigma^{-1} \mu_e \\ &= \frac{\Sigma^{-1} \mu_e}{\mathbf{1}' \Sigma^{-1} \mu_e}. \end{aligned}$$

We can see that θ_T does not depend on the risk aversion coefficient a , which is what the **separation theorem** states.

Define $s = \mathbf{1}' \Sigma^{-1} \mu_e$, the denominator of θ_T , and assume $s > 0$, so that the rescaling does not flip the sign of the weights. Then the **optimal portfolio** is a scaled copy of θ_T , namely $\theta^* = \frac{s}{a} \theta_T$. Risk aversion a only sets the total weight $\frac{s}{a}$ on the risky assets. It never changes the mix among them.

(ii) The Highest Sharpe Ratio

Next, we prove that θ_T is the **super-efficient portfolio**, the one with the maximal **Sharpe ratio**.

Given any risky portfolio $\theta \in \mathbb{R}^N$ with $\theta' \mathbf{1} = 1$, the return of the portfolio is $R_\theta = \theta' R \in \mathbb{R}$. Its expectation is $\mathbb{E}[R_\theta] = \mathbb{E}[\theta' R] = \theta' \mathbb{E}[R]$, and its variance is $\text{Var}[R_\theta] = \text{Var}[\theta' R] = \theta' \Sigma \theta$. Consequently, the Sharpe ratio $S(\theta)$ is

$$\begin{aligned} S(\theta) &= \frac{\mathbb{E}[R_\theta] - R_f}{\sqrt{\text{Var}[R_\theta]}} \\ &= \frac{\mathbb{E}[\theta' R] - R_f}{\sqrt{\text{Var}[\theta' R]}} \\ &= \frac{\theta' \mathbb{E}[R] - R_f}{\sqrt{\theta' \Sigma \theta}} \\ &= \frac{\theta' \mathbb{E}[R] - \theta' \mathbf{1} R_f}{\sqrt{\theta' \Sigma \theta}} \\ &= \frac{\theta' \mu_e}{\sqrt{\theta' \Sigma \theta}}. \end{aligned}$$

Therefore, using $s > 0$ in the last step,

$$S(\theta_T) = \frac{(\mu_e' \Sigma^{-1} \mu_e) / \mathbf{1}' \Sigma^{-1} \mu_e}{\sqrt{(\mu_e' \Sigma^{-1} \mu_e) / (\mathbf{1}' \Sigma^{-1} \mu_e)^2}} = \sqrt{\mu_e' \Sigma^{-1} \mu_e}.$$

Apply the [Cauchy–Schwarz inequality](#) to $\Sigma^{\frac{1}{2}} \theta$ and $\Sigma^{-\frac{1}{2}} \mu_e$:

$$\theta' \mu_e = (\Sigma^{\frac{1}{2}} \theta)' (\Sigma^{-\frac{1}{2}} \mu_e) \leq \sqrt{\theta' \Sigma \theta} \sqrt{\mu_e' \Sigma^{-1} \mu_e},$$

so $S(\theta) \leq \sqrt{\mu_e' \Sigma^{-1} \mu_e} = S(\theta_T)$. Consequently, θ_T is the super-efficient portfolio.

On the plot of expected return against standard deviation, $S(\theta)$ is the slope of the line from $(0, R_f)$ to the portfolio. The steepest such line that still reaches the [risky frontier](#) touches it at θ_T . That line is the [capital market line](#), and θ_T is the [super-efficient portfolio](#) where it touches.

4. Market Clearing and the Market Portfolio

(i) Dollar Demand

Suppose there are I investors with the same beliefs $R \sim N(\mathbb{E}[R], \Sigma)$, where investor i has CARA risk aversion a_i and starting wealth $W_{0,i}$. Let $x_i \in \mathbb{R}^N$ be the investor's [dollar holdings](#) of the risky assets, so the rest $W_{0,i} - \mathbf{1}' x_i$ sits in the risk-free asset and

$$W_i = x_i' R + (W_{0,i} - \mathbf{1}' x_i) R_f = W_{0,i} R_f + x_i' (R - R_f \mathbf{1})$$

This has mean $W_{0,i}R_f + x'_i\mu_e$ and variance $x'_i\Sigma x_i$. The constant $W_{0,i}R_f$ only multiplies the objective by the positive factor $\exp(-a_i W_{0,i}R_f)$, so it does not move the maximizer, and the **same FOC** gives

$$x_i = \frac{1}{a_i} \Sigma^{-1} \mu_e.$$

The starting wealth drops out of the dollar demand. This is a feature of CARA utility, and it is what makes the demands easy to add up.

(ii) Market Clearing

Let $m \in \mathbb{R}^N$ be the **supply** of the risky assets: the total market value of each asset, that is, its price times its shares outstanding. The issuers fix the shares outstanding but not the prices. Market clearing sets the prices, so that in equilibrium the **total demand** $\sum_{i=1}^I x_i$ equals this supply:

$$\sum_{i=1}^I x_i = \left(\sum_{i=1}^I \frac{1}{a_i} \right) \Sigma^{-1} \mu_e = m \Rightarrow \Sigma^{-1} \mu_e = Am, \quad A = \left(\sum_{i=1}^I \frac{1}{a_i} \right)^{-1},$$

where A is the aggregate absolute risk aversion of all investors.

Moreover, this equation is the condition that the **equilibrium prices** satisfy: at those prices, $\mu_e = A\Sigma m$, and investors want to hold exactly the shares outstanding.

(iii) The Super-Efficient Portfolio is the Market Portfolio

Definition (Market Portfolio)

The market portfolio holds each risky asset in proportion to its market value, $\theta_M = \frac{m}{\mathbf{1}'m}$, and its return is $R_{\theta_M} = \theta'_M R$.

Substituting $\Sigma^{-1} \mu_e = Am$ into the **super-efficient portfolio** gives

$$\theta_T = \frac{Am}{A\mathbf{1}'m} = \theta_M,$$

so in equilibrium the super-efficient portfolio is the **market portfolio**.

If investors held different beliefs about μ_e or Σ , they would hold different super-efficient portfolios, and the market portfolio would be a mixture of them rather than any one investor's θ_T . The same goes for investors who face different constraints, such as limits on short-selling.

5. The CAPM Equation

From [market clearing](#),

$$\mu_e = A\Sigma m = A(\mathbf{1}'m)\Sigma\theta_M.$$

By the bilinearity of covariance, the j -th entry of $\Sigma\theta_M$ is

$\text{Cov}[R_j, \theta_M' R] = \text{Cov}[R_j, R_{\theta_M}]$, so for each risky asset j

$$\mathbb{E}[R_j] - R_f = A(\mathbf{1}'m) \text{Cov}[R_j, R_{\theta_M}].$$

Multiply the market-clearing [equation](#) above on the left by θ_M' . Since $\theta_M' \mathbf{1} = 1$, the left side becomes $\theta_M' \mathbb{E}[R] - R_f = \mathbb{E}[R_{\theta_M}] - R_f$, and since $\theta_M' \Sigma \theta_M = \text{Var}[R_{\theta_M}]$, the same relation holds for the market itself:

$$\mathbb{E}[R_{\theta_M}] - R_f = A(\mathbf{1}'m) \text{Var}[R_{\theta_M}].$$

Dividing the first [equation](#) by the [second](#) gives the CAPM.

Theorem (Capital Asset Pricing Model (CAPM))

For every risky asset j ,

$$\mathbb{E}[R_j] - R_f = \beta_j (\mathbb{E}[R_{\theta_M}] - R_f), \quad \beta_j = \frac{\text{Cov}[R_j, R_{\theta_M}]}{\text{Var}[R_{\theta_M}]}.$$

6. Equilibrium Prices

The derivation above takes the return distribution as given, which already fixes the prices, since the return $R_j = \frac{X_j}{p_j}$ divides the payoff X_j of asset j by its price p_j . To solve for the prices, write the model in payoffs instead: the shares outstanding $n \in \mathbb{R}^N$ and the end-of-period payoffs $X \sim N(\mathbb{E}[X], \Omega)$ are given, and the prices $p \in \mathbb{R}^N$ are unknown.

(i) Share Demand

Investor i buys $z_i \in \mathbb{R}^N$ shares and puts the rest of their wealth in the risk-free asset:

$$W_i = (W_{0,i} - z_i' p) R_f + z_i' X = W_{0,i} R_f + z_i' (X - R_f p).$$

This has the same form as the [dollar demand](#) problem, with $X - R_f p$ in place of $R - R_f \mathbf{1}$, so the same FOC gives

$$z_i = \frac{1}{a_i} \Omega^{-1} (\mathbb{E}[X] - R_f p).$$

(ii) Market Clearing

All shares must be held, $\sum_{i=1}^I z_i = n$. With $\frac{1}{A} = \sum_{i=1}^I \frac{1}{a_i}$, this gives

$$\frac{1}{A} \Omega^{-1} (\mathbb{E}[X] - R_f p) = n \quad \Rightarrow \quad p = \frac{\mathbb{E}[X] - A \Omega n}{R_f}.$$